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Twisted Grosse-Wulkenhaar $\phi^{\star 4}$ model: dynamical noncommutativity and Noether currents

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Abstract

This paper addresses the computation of Noether currrents for the renormalizable Grosse-Wulkenhaar (GW) $\phi^{\star 4}$ model subjected to a dynamical noncomutativity realized through a twisted Moyal product. The noncommutative (NC) energy-momentum tensor (EMT), angular momentum tensor (AMT) and the dilatation current (DC) are explicitly derived. The breaking of translation and rotation invariances has been avoided via a constraint equation.

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1 Introduction

Most of different settings for noncommutative (NC) field theories [1] - [19] are based on a Moyal space R^D_Θ, a deformed D-dimensional space endowed with a constant Moyal $\star-$bracket of coordinate functions

$$[x^\mu, x^\nu]_\star = i\Theta^{\mu\nu} \tag{1}$$

where Θ is a $D \times D$ non-degenerate skew-symmetric matrix (which requires D even), usually chosen in the form

$$\Theta = \begin{pmatrix} 0 & \Theta_1 & & & & & & \\ -\Theta_1 & 0 & & & & 0 & & \\ & & 0 & \Theta_2 & & & & \\ & & -\Theta_2 & 0 & & & & \\ & & & & \vdots & \vdots & & \\ & & & & \vdots & \vdots & & \\ & & 0 & & & & 0 & \Theta_{\frac{D}{2}} \\ & & & & & & -\Theta_{\frac{D}{2}} & 0 \end{pmatrix} \tag{2}$$

where $\Theta_j \in \mathbb{R}$, $j = 1, 2, \cdots, \frac{D}{2}$, have dimension[2] of length square, ($[\Theta_j] = [L]^2$), D denoting the spacetime dimension. The corresponding product of functions is the associative, noncommutative Moyal-Groenewold-Weyl product, simply called hereafter Moyal product or $\star$-product defined by

$$(f \star g)(x) = \mathrm{m}\left\{e^{i\frac{\Theta^{\rho\sigma}}{2}\partial_\rho \otimes \partial_\sigma} f(x) \otimes g(x)\right\},\ x \in \mathrm{R}^D_\Theta, \qquad \forall f, g \in \mathcal{S}(\mathrm{R}^D_\Theta) \tag{3}$$

m is the ordinary multiplication of functions and $\mathcal{S}(\mathrm{R}^D_\Theta)$ - the space of suitable Schwartzian functions. For more details, see [11]-[14]. Such a noncommutative geometry possesses the specific pathology to break both the Lorentz invariance by the presence of $\Theta^{\mu\nu}$, as $[x^\mu, x^\nu]_\star = i\Theta^{\mu\nu}$ is not generally invariant under rotation, and the local character of the theory due to infinite time derivatives. There result energy momentum tensors (EMTs) which are not locally conserved, not traceless in the massless situation and, not symmetric and not gauge invariant in gauge theories. A number of works exist in attempts to achieve regularization for the NC EMT which then becomes symmetric albeit not locally conserved. Further improvement of this quantity by

[2]Units such that $\hbar = 1 = c$ are used throughout.

usual algebraic tricks breaks its symmetry, (see [13] and references therein). Therefore, the property of nonlocal conservation of angular momentum is not a priori proscribed.

Recently, Paolo Aschieri *et al* [1] introduced a so-called dynamical noncommutativity to investigate Noether currents in an ordinary nonrenormalizable twisted $\phi^{\star 4}$ theory. This work addresses questions of the applicability of such a formalism on the new class of renormalizable NC field theories (NCRFT) built on the Grosse and Wulkenhaar (GW) $\phi^{\star 4}$ scalar field model defined in Euclidean space-time by the action functional [12]

$$\begin{aligned} S^{\Omega}_{\star}[\phi] &= \int \mathrm{d}^D x \Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{m^2}{2}\phi\star\phi \\ &\quad +\frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big), \end{aligned} \tag{4}$$

where $\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu}x^\nu$ and $S^{\Omega}_{\star}[\phi]$ is covariant under Langmann-Szabo duality [19]. Ω and λ are dimensionless parameters. Generalizing the Moyal $\star$-product (3) under the form

$$(f\star g)(x) = m\Big\{e^{i\frac{\Theta^{ab}}{2}X_a\otimes X_b}f(x)\otimes g(x)\Big\} =: e^{\Delta}(f,g) \tag{5}$$

where $X_a = e^\mu_a(x)\partial_\mu$ is a commuting vector fields, the commutation relation becomes $[x^\mu, x^\nu]_\star = i\Theta^{ab}e^\mu_a(x)e^\nu_b(x) =: i\widetilde{\Theta}^{\mu\nu}(x)$, engendering a twisted scalar field theory where e^μ_a, and hence the $\star$ product itself, appear dynamical. See Appendix for useful relations concerning this generalized product. The condition $[X_a, X_b] = 0$ implies constraints on e^μ_a, namely $e^\nu_{[a}\partial_\nu e^\mu_{b]} = 0$, that can be solved off-shell in terms of D scalar fields ϕ^a, (see [1] and [2]). Supposing that the square matrix e^μ_a has an inverse e^a_μ everywhere, so that the X_a are linearly independent, then the above condition becomes $\partial_{[\mu}e^a_{\nu]} = 0$ which is satisfied by $e^a_\nu = \partial_\nu\phi^a$. Besides, the Leibniz rule extends to the commuting fields X_a as follows: $X_a(f\star g) = (X_af)\star g + f\star(X_ag)$.

This paper is organized as follows. In Section 2, we derive the field equations of motion and provide with the explicit computation of noncommutative energy momentum tensor (NC EMT), angular momentum tensor (AMT) and dilatation current (DC). Furtheremore, we proceed to the symmetry analysis including the translation, rotation and dilatation transformations and compute the conserved currents. Finally, we end with some concluding remarks in Section 3.

2 Twisted Grosse-Wulkenhaar model: Noether currents

The generalized NC GW Lagrangian action corresponding to (4) can be written as:

$$\begin{aligned} \mathcal{S}^{\Omega}_{\star}[\phi] &= \int ed^D x \left(\mathcal{L}^{\Omega}_{\star} \star e^{-1}\right) \\ &= \int e\mathrm{d}^D x \Big\{\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \\ &\quad + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{1}{2}\partial_\mu\phi_a\star\partial^\mu\phi^a\Big\}\star e^{-1}, \end{aligned} \tag{6}$$

$$\text{where} \quad e = \mathrm{det} e^a_\mu$$

from which the peculiar Euler Lagrange equations of motion can be readily derived by direct application of the variational principle. There results the following.

i) From the ϕ-variation, the equation of motion of the field ϕ is expressed as:

$$\begin{aligned} \mathcal{E}_\phi &= -\frac{1}{2}\partial_\sigma\Big(e\{\partial^\sigma\phi, e^{-1}\}_\star\Big) + \frac{m^2}{2}e\{\phi, e^{-1}\}_\star + \frac{\lambda}{4!}e\{\phi\star\phi, \{\phi, e^{-1}\}_\star\}_\star \\ &+ \frac{\Omega^2}{8}e\{\tilde{x}, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star = 0. \end{aligned} \tag{7}$$

In the commutative limit $\Theta \to 0$, the equation (7) becomes the usual ϕ^4 field equation of motion

$$\Box\phi - m^2\phi - \frac{\lambda}{3!}\phi^3 = 0. \tag{8}$$

The current $\mathcal{K}^\sigma$ is determined by the expression

$$\mathcal{K}^\sigma = \mathcal{K}^\sigma(0) + \mathcal{K}^\sigma(m^2) + \mathcal{K}^\sigma(\lambda) + \mathcal{K}^\sigma(\Omega^2), \tag{9}$$

where the four main contributions are induced by the velocity field

$$\begin{aligned} \mathcal{K}^\sigma(0) &= \frac{e\delta\phi}{2}.\{\partial^\sigma\phi, e^{-1}\}_\star + ee^\sigma_b\Big[T(\Delta)\Big(\delta\partial_\mu\phi, \frac{\widetilde{X}^b}{2}\{\partial^\mu\phi, e^{-1}\}_\star\Big) \\ &\quad + S(\Delta)\Big(\partial_\mu\phi, \widetilde{X}^b(\partial^\mu\delta\phi\star e^{-1})\Big)\Big], \end{aligned} \tag{10}$$

the mass term

$$
\begin{aligned}
\mathcal{K}^\sigma(m^2) &= ee^\sigma_b\Big[\frac{m^2}{2}T(\Delta)\Big(\delta\phi, \widetilde{X}^b\{\phi, e^{-1}\}_\star\Big) \\
&\quad +m^2 S(\Delta)\Big(\phi, \widetilde{X}^b(\delta\phi\star e^{-1})\Big)\Big],
\end{aligned}
\tag{11}
$$

the $\phi^{\star 4}$ interaction

$$
\begin{aligned}
\mathcal{K}^\sigma(\lambda) &= ee^\sigma_b\Big[\frac{\lambda}{4!}T(\Delta)\Big(\delta\phi, \widetilde{X}^b\{\phi\star\phi, \{\phi, e^{-1}\}_\star\}_\star\Big) \\
&\quad +\frac{\lambda}{12}S(\Delta)\Big(\phi, \widetilde{X}^b(\delta\phi\star\phi\star\phi\star e^{-1})\Big) \\
&\quad +\frac{\lambda}{12}S(\Delta)\Big(\phi\star\phi, \widetilde{X}^b(\delta\phi\star\phi\star e^{-1})\Big) \\
&\quad +\frac{\lambda}{12}S(\Delta)\Big(\phi\star\phi\star\phi, \widetilde{X}^b(\delta\phi\star e^{-1})\Big)\Big]
\end{aligned}
\tag{12}
$$

and the GW harmonic interaction

$$
\begin{aligned}
\mathcal{K}^\sigma(\Omega^2) &= ee^\sigma_b\Big[\frac{\Omega^2}{8}T(\Delta)\Big(\delta\phi, \widetilde{X}^b\{\tilde{x}, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star\Big) \\
&\quad +\frac{\Omega^2}{4}S(\Delta)\Big(\tilde{x}, \widetilde{X}^b(\delta\phi\star\{\tilde{x},\phi\}_\star\star e^{-1})\Big) \\
&\quad +\frac{\Omega^2}{4}S(\Delta)\Big(\{\tilde{x}, \phi\star\tilde{x}\}_\star, X^b(\delta\phi\star e^{-1})\Big) \\
&\quad +\frac{\Omega^2}{4}S(\Delta)\Big(\{\phi, \tilde{x}\}_\star, \widetilde{X}^b(\delta\phi\star\tilde{x}\star e^{-1})\Big)\Big],
\end{aligned}
\tag{13}
$$

respectively. See Appendix for definitions and notation.

ii) From the ϕ^c-variation, after tedious algebraic transformations, we get the following field equation:

$$
\begin{aligned}
\mathcal{E}_{(\phi,\phi^c)} &= e\Big[\frac{1}{e}X_c(\mathcal{L}^\Omega_\star) - (X_c\phi)\Big(\frac{m^2}{2}\{\phi, e^{-1}\}_\star + \frac{\lambda}{4!}\{\phi\star\phi, \{\phi, e^{-1}\}_\star\}_\star \\
&\quad +\frac{\Omega^2}{2}\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star\Big) - \frac{\Omega^2}{2}\phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star \\
&\quad -\frac{1}{2}X_c\partial_\mu\phi.\{\partial^\mu\phi, e^{-1}\}_\star - \frac{1}{2}X_c\partial_\mu\phi_a.\{\partial^\mu\phi^a, e^{-1}\}_\star \\
&\quad -\frac{1}{e}\partial_\mu\Big(\frac{e}{2}\{\partial^\mu\phi_c, e^{-1}\}_\star\Big] = 0.
\end{aligned}
\tag{14}
$$

Using the identities $\tilde{x}_\mu \star \phi = \tilde{x}_\mu\phi + i\partial_\mu\phi$ and $\phi \star \tilde{x}_\mu = \tilde{x}_\mu\phi - i\partial_\mu\phi$ implying $\tilde{x}\phi = \frac{1}{2}\{\tilde{x},\phi\}_\star$, we can deduce that $\frac{\Omega^2}{2}\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star = \frac{\Omega^2}{8}\{\tilde{x},\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star$, and the equation of motion takes the form

$$\begin{aligned}
\mathcal{E}_{(\phi,\phi^c)} &= -X_c\phi\mathcal{E}_\phi + X_c\mathcal{L}^\Omega_\star - \frac{1}{2}X_c\phi\partial_\mu\Big(e\{\partial^\mu\phi, e^{-1}\}_\star\Big) \\
&\quad -e\frac{\Omega^2}{2}\phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star - \frac{e}{2}X_c\partial_\mu\phi.\{\partial^\mu\phi, e^{-1}\}_\star \\
&\quad -\frac{e}{2}X_c\partial_\mu\phi_a.\{\partial^\mu\phi^a, e^{-1}\}_\star - \partial_\mu\Big(\frac{e}{2}\{\partial^\mu\phi_c, e^{-1}\}_\star\Big) \\
&= -X_c\phi\mathcal{E}_\phi - \mathcal{E}_{\phi^c} = 0, \qquad (15)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{E}_{\phi^c} &= -X_c\mathcal{L}^\Omega_\star + \frac{1}{2}X_c\phi\partial_\mu\Big(e\{\partial^\mu\phi, e^{-1}\}_\star\Big) + e\frac{\Omega^2}{2}\phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star \\
&\quad +\frac{e}{2}X_c\partial_\mu\phi.\{\partial^\mu\phi, e^{-1}\}_\star + \frac{e}{2}X_c\partial_\mu\phi_a.\{\partial^\mu\phi^a, e^{-1}\}_\star \\
&\quad +\partial_\mu\Big(\frac{e}{2}\{\partial^\mu\phi_c, e^{-1}\}_\star\Big) \qquad (16)
\end{aligned}$$

with

$$\frac{\Omega^2}{2}\phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star = \frac{\Omega^2}{8}X_c\tilde{x}.\{\phi,\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star.$$

One can immediately show that, as expected from [1], when ϕ is on shell (i.e. $\mathcal{E}_\phi = 0$, the ϕ^c field equation of motion simply reduces to $\mathcal{E}_{\phi^c} = 0$, and in the commutative limit, we get $\Box\phi^c = 0$ as it should. Besides, the field equations (7) and (16) are satisfied by the vacuum solution $\phi = 0$, $e^a_\mu = \partial_\mu\phi^a = \delta^a_\mu$ corresponding to the usual Moyal product. The field ϕ acts as a source for the noncommutativity field ϕ^c.

The current $\mathcal{J}^\sigma$ is given by

$$\mathcal{J}^\sigma = \mathcal{J}^\sigma(0) + \mathcal{J}^\sigma(m^2) + \mathcal{J}^\sigma(\lambda) + \mathcal{J}^\sigma(\Omega^2) \qquad (17)$$

where the contributions engendered by the velocity field, the mass term, the $\phi^{\star 4}$ interaction and the GW harmonic interaction source are, respectively, expressed as

$$
\begin{aligned}
\mathcal{J}^\sigma(0) &= \frac{1}{2}e\delta\phi^a\{\partial^\sigma\phi_a, e^{-1}\}_\star \\
&+ee^\sigma_b\Big\{\frac{1}{2}\Big[-T(\Delta)\Big(\delta\phi^c X_c\partial_\mu\phi_a, \widetilde{X}^b\{\partial^\mu\phi^a, e^{-1}\}_\star\Big) \\
&-2S(\Delta)\Big(\partial^\mu\phi_a, \widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi^a)\star e^{-1})\Big) \\
&+2S(\Delta)\Big(\partial_\mu\phi_a, \widetilde{X}^b(\partial^\mu\delta\phi^a\star e^{-1})\Big) \\
&+2T(\Delta)\Big(\delta\partial_\mu\phi_a, \frac{\widetilde{X}^b}{2}\{\partial^\mu\phi^a, e^{-1}\}_\star\Big)\Big] \\
&+\frac{1}{2}\Big[-T(\Delta)\Big(\delta\phi^c X_c\partial_\mu\phi, \widetilde{X}^b\{\partial^\mu\phi, e^{-1}\}_\star\Big) \\
&-2S(\Delta)\Big(\partial^\mu\phi, \widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi)\star e^{-1})\Big)\Big] \\
&-\mathcal{L}^\Omega_\star(0)\star(\delta\phi^b e^{-1}) + \delta\phi^b(\mathcal{L}^\Omega_\star(0)\star e^{-1}) \\
&+T(\Delta)\Big(X_c(\mathcal{L}^\Omega_\star(0)), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big\}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\mathcal{J}^\sigma(m^2) &= ee^\sigma_b\Big\{\frac{m^2}{2}\Big[-T(\Delta)\Big(\delta\phi^a(X_a\phi), \widetilde{X}^b\{\phi, e^{-1}\}\Big) \\
&+2S(\Delta)\Big(\delta\phi^a(X_a\phi)\star e^{-1}, \widetilde{X}^b\phi\Big)\Big] \\
&-\mathcal{L}^\Omega_\star(m^2)\star(\delta\phi^b e^{-1}) + \delta\phi^b(\mathcal{L}^\Omega_\star(m^2)\star e^{-1}) \\
&+T(\Delta)\Big(X_c(\mathcal{L}^\Omega_\star(m^2)), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big\}
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\mathcal{J}^\sigma(\lambda) &= ee^\sigma_b\Big\{\frac{\lambda}{4!}\Big[-T(\Delta)\Big(\delta\phi^c X_c\phi, \widetilde{X}^b\{\phi\star\phi, \{\phi, e^{-1}\}_\star\}_\star\Big) \\
&-2S(\Delta)\Big(\phi, \widetilde{X}^b((\delta\phi^c X_c\phi)\star\phi\star\phi\star e^{-1})\Big) \\
&-2S(\Delta)\Big(\phi\star\phi, \widetilde{X}^b((\delta\phi^c X_c\phi)\star\phi\star e^{-1})\Big) \\
&-2S(\Delta)\Big(\phi\star\phi\star\phi, \widetilde{X}^b((\delta\phi^c X_c\phi)\star e^{-1})\Big)\Big] \\
&-\mathcal{L}^\Omega_\star(\lambda)\star(\delta\phi^b e^{-1}) + \delta\phi^b(\mathcal{L}^\Omega_\star(\lambda)\star e^{-1}) \\
&+T(\Delta)\Big(X_c(\mathcal{L}^\Omega_\star(\lambda)), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big\}
\end{aligned} \tag{20}
$$

$$
\mathcal{J}^\sigma(\Omega^2) = ee^\sigma_b\Big\{\frac{\Omega^2}{2}\Big[-T(\Delta)\Big(\delta\phi^c X_c(\tilde{x}\phi), \widetilde{X}^b\{\tilde{x}\phi, e^{-1}\}_\star\Big)
$$

$$
\begin{aligned}
&-2S(\Delta)\Big(\tilde{x}\phi, \widetilde{X}^b((\delta\phi^c X_c(\tilde{x}\phi)) \star e^{-1})\Big)\Big] \\
&-\mathcal{L}_\star^\Omega(\Omega^2) \star (\delta\phi^b e^{-1}) + \delta\phi^b(\mathcal{L}_\star^\Omega(\Omega^2) \star e^{-1}) \\
&+T(\Delta)\Big(X_c(\mathcal{L}_\star^\Omega(\Omega^2)), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big\},
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
&\mathcal{L}_\star^\Omega(0) = \frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}\partial_\mu\phi_a \star \partial^\mu\phi^a, \quad \mathcal{L}_\star^\Omega(m^2) = \frac{m^2}{2}\phi \star \phi, \\
&\mathcal{L}_\star^\Omega(\lambda) = \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi, \quad \mathcal{L}_\star^\Omega(\Omega^2) = \frac{\Omega^2}{2}(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi).
\end{aligned} \tag{22}
$$

Let us now deal with the symmetry analysis and deduce the conserved currents. Performing the functional variation of the fields and coordinate transformation

$$
\phi'(x) = \phi(x) + \delta\phi(x), \quad \phi'^c(x) = \phi^c(x) + \delta\phi^c(x), \quad x'^\mu = x^\mu + \epsilon^\mu \tag{23}
$$

and using $\mathrm{d}^D x' = [1 + \partial_\mu\epsilon^\mu + \mathbf{O}(\epsilon^2)]\mathrm{d}^D x$ lead to the following variation of the action, to first order in $\delta\phi(x), \delta\phi^c(x), \tilde{x}$ and ϵ^μ:

$$
\begin{aligned}
\delta\mathcal{S}_\star^\Omega &= \int e\mathrm{d}^D x\Big\{\Big|\frac{\partial x'}{\partial x}\Big| \star (\mathcal{L}_\star'^\Omega \star e^{-1})\Big\} - \int e\mathrm{d}^D x\,(\mathcal{L}_\star^\Omega \star e^{-1}) \\
&= \int \mathrm{d}^D x\Big\{\delta\Big((\mathcal{L}_\star^\Omega \star e^{-1})e\Big) + \partial_\mu\epsilon^\mu \star \Big((\mathcal{L}_\star^\Omega \star e^{-1})e\Big)\Big\} \\
&= \int \mathrm{d}^D x\,\Big\{\delta_\phi\Big((\mathcal{L}_\star^\Omega \star e^{-1})e\Big) + \delta_{\phi^c}\Big((\mathcal{L}_\star^\Omega \star e^{-1})e\Big) \\
&\quad +\delta_{\tilde{x}}\Big(\mathcal{L}_\star^\Omega \star e^{-1})e\Big) + \epsilon^\mu \star \partial_\mu[(\mathcal{L}_\star^\Omega \star e^{-1})e] \\
&\quad +\partial_\mu\epsilon^\mu \star (\mathcal{L}_\star^\Omega \star e^{-1})e\Big\}.
\end{aligned} \tag{24}
$$

On shell, and integrated on a manifold M (so that the total derivative terms do not disappear), we get:

$$
\delta\mathcal{S}_\star^\Omega = \int_M \mathrm{d}^D x\,\partial_\sigma\Big[\mathcal{K}^\sigma + \mathcal{J}^\sigma + \mathcal{R}^\sigma + \epsilon^\sigma \star \Big((\mathcal{L}_\star^\Omega \star e^{-1})e\Big)\Big] \tag{25}
$$

where $\mathcal{R}^\sigma$ is defined as follows:

$$
\mathcal{R}^\sigma \;=\; \frac{\Omega^2}{8}ee_b^\sigma\Big\{T(\Delta)\Big(\delta\tilde{x}, \widetilde{X}^b\{\phi, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star\Big)
$$

$$+2S(\Delta)\Big(\{\tilde{x},\phi\}_\star,\widetilde{X}^b(\delta\tilde{x}\star\phi\star e^{-1})\Big)$$
$$+2S(\Delta)\Big(\{\phi,\tilde{x}\star\phi\}_\star,\widetilde{X}^b(\delta\tilde{x}\star e^{-1})\Big)$$
$$+2S(\Delta)\Big(\phi,\widetilde{X}^b(\delta\tilde{x}\star\{\tilde{x},\phi\}_\star\star e^{-1})\Big)\Big\}. \tag{26}$$

Therefore the current $\mathcal{J}^\sigma$ reads

$$\begin{aligned}
\mathcal{J}^\sigma \;=\;& \mathcal{K}^\sigma(\delta\phi\to-\delta\phi^c X_c\phi)+\mathcal{R}^\sigma(\delta\tilde{x}\to-\delta\phi^c X_c\tilde{x})\\
&+\frac{e\delta\phi^c}{2}X_c\phi.\{\partial^\sigma\phi,e^{-1}\}_\star+\frac{e\delta\phi^c}{2}.\{\partial^\sigma\phi_c,e^{-1}\}_\star\\
&+ee_b^\sigma\Big\{-\mathcal{L}_\star^\Omega\star(\delta\phi^b e^{-1})+\delta\phi^b(\mathcal{L}_\star^\Omega\star e^{-1})\\
&+T(\Delta)\Big(X_c(\mathcal{L}_\star^\Omega),\widetilde{X}^b(\delta\phi^c e^{-1})\Big)\\
&+\frac{1}{2}T(\Delta)\Big(\partial_\mu(\delta\phi^c e_c^\rho)\partial_\rho\phi,\widetilde{X}^b\{\partial^\mu\phi,e^{-1}\}_\star\Big)\\
&+S(\Delta)\Big(\partial_\mu\phi,\widetilde{X}^b((\partial_\mu(\delta\phi^c e_c^\rho)\partial_\rho\phi)\star e^{-1})\Big)\Big\}\\
&+\frac{1}{2}ee_b^\sigma\Big\{-T(\Delta)\Big(\delta\phi^c X_c\partial_\mu\phi_a,\widetilde{X}^b\{\partial^\mu\phi^a,e^{-1}\}_\star\Big)\\
&-2S(\Delta)\Big(\partial^\mu\phi_a,\widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi^a)\star e^{-1})\Big)\\
&+2S(\Delta)\Big(\partial_\mu\phi_a,\widetilde{X}^b(\partial^\mu\delta\phi^a\star e^{-1})\Big)\\
&+T(\Delta)\Big(\partial_\mu\delta\phi_a,\widetilde{X}^b\{\partial^\mu\phi^a,e^{-1}\}_\star\Big)\Big\}.
\end{aligned}\tag{27}$$

$\mathcal{K}^\sigma$ keeps the previous defined expression. In contrary to the result in [1] for ordinary $\phi_\star^4$ theory, the twisted GW action is not invariant under global translation. Now imposing the constraint $\frac{\delta\mathcal{S}_\star^\Omega}{\delta\tilde{x}}=0$ giving

$$e\frac{\Omega^2}{8}\{\phi,\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star=0, \tag{28}$$

coupled to the transformations

$$\delta\phi=-\epsilon^\nu\partial_\nu\phi,\quad \delta\phi^c=-\epsilon^\nu\partial_\nu\phi^c,\quad \epsilon^\nu=\text{constant} \tag{29}$$

that we subtitute into (25) and taking into account $e_\nu^a=\partial_\nu\phi^a$, we infer from the relation

$$0=\delta\mathcal{S}_\star^\Omega=\int_M \mathrm{d}^D x\,\epsilon^\nu\partial_\mu\mathcal{T}_\nu^\mu \tag{30}$$

the EMT

$$
\begin{aligned}
\mathcal{T}^{\mu}_{\nu} \;=\; & -\frac{e}{2}(\partial_\nu\phi)\{\partial^\mu\phi, e^{-1}\}_\star - \frac{e}{2}(\partial_\nu\phi_c)\{\partial^\mu\phi^c, e^{-1}\}_\star \\
& + e e^\mu_b \Big\{ \mathcal{L}^\Omega_\star \star (e^{-1}\partial_\nu\phi^b) + T(\Delta)\Big(X_c\mathcal{L}^\Omega_\star, \widetilde{X}^b(e^{-1}\partial_\nu\phi^c)\Big) \\
& + \Omega^2\Theta^{-1}_{\gamma\nu}\Big[S(\Delta)\Big(\{\tilde{x}^\gamma, \phi\}_\star, \widetilde{X}^b(\phi\star e^{-1})\Big) \\
& + S(\Delta)\Big(\{\phi, \tilde{x}^\gamma \star \phi\}_\star, \widetilde{X}^b(e^{-1})\Big) \\
& + S(\Delta)\Big(\phi, \widetilde{X}^b\{\tilde{x}^\gamma, \phi\}_\star \star e^{-1}\Big)\Big]\Big\}.
\end{aligned} \tag{31}
$$

This tensor is neither symmetric nor locally conserved. In the case of standard Moyal product, it reduces to the NC EMT computed in [13] and its regularization can be worked out in the same way as done in that work. Similary, the transformation

$$
\delta\phi = -\epsilon^\nu\partial_\nu\phi = -\epsilon^{\nu\rho}x_\rho\partial_\nu\phi, \quad \delta\phi^c = -\epsilon^\nu\partial_\nu\phi^c = -\epsilon^{\nu\rho}x_\rho\partial_\nu\phi^c, \quad \epsilon^\nu = \epsilon^{\nu\rho}x_\rho \tag{32}
$$

with $\epsilon^{\nu\rho}$ an infinitesimal constant skew symmetric Lorentz parameter, and $\epsilon^{\nu\rho}x_{[\nu}\partial_{\rho]}\phi = -2\epsilon^{\nu\rho}x_\rho\partial_\nu\phi$, substituted into (25) yields

$$
0 = \delta\mathcal{S}^\Omega_\star = \int_M \mathrm{d}^D x\, \epsilon^{\nu\rho}\partial_\mu\mathcal{M}^\mu_{\nu\rho}, \tag{33}
$$

which affords the AMT as

$$
\begin{aligned}
\mathcal{M}^\mu_{\nu\rho} \;=\; & \frac{e}{4}x_{[\nu}\partial_{\rho]}\phi\{\partial^\mu\phi, e^{-1}\}_\star + \frac{e}{4}x_{[\nu}\partial_{\rho]}\phi_c\{\partial^\mu\phi^c, e^{-1}\}_\star \\
& - \frac{ee^\mu_b}{2}\Big(\mathcal{L}^\Omega_\star \star (e^{-1}x_{[\nu}\partial_{\rho]}\phi^b)\Big) \\
& + \frac{ee^\mu_b}{2}\Big\{T(\Delta)\Big(X_c\mathcal{L}^\Omega_\star, \widetilde{X}^b(e^{-1}x_{[\nu}\partial_{\rho]}\phi^c)\Big) \\
& - T(\Delta)\Big(\partial_{[\nu}\phi, \frac{1}{2}\widetilde{X}^b(\{\partial_{\rho]}\phi, e^{-1}\}_\star)\Big) \\
& - T(\Delta)\Big(\partial_{[\nu}\phi^d, \frac{1}{2}\widetilde{X}^b(\{\partial_{\rho]}\phi_d, e^{-1}\}_\star)\Big) \\
& + S(\Delta)\Big(\partial_{[\nu}\phi, \widetilde{X}^b(\partial_{\rho]}\phi \star e^{-1})\Big) \\
& + S(\Delta)\Big(\partial_{[\nu}\phi_d, \widetilde{X}^b(\partial_{\rho]}\phi^d \star e^{-1})\Big) \\
& - \frac{\Omega^2}{4}\Theta^{-1}_{\gamma[\nu}\Big[T(\Delta)\Big(x_{\rho]}, \widetilde{X}^b(\{\phi, \{e^{-1}, \{\tilde{x}^\gamma, \phi\}_\star\}_\star\}_\star)\Big) \\
& + 2S(\Delta)\Big(\{\tilde{x}^\gamma, \phi\}_\star, \widetilde{X}^b(x_{\rho]} \star \phi \star e^{-1})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+2S(\Delta)\Big(\{\phi, \tilde{x}^{\gamma}\star\phi\}_\star, \widetilde{X}^b(x_{\rho]}\star e^{-1})\Big)\\
&+2S(\Delta)\Big(\phi, \widetilde{X}^b(x_{\rho]}\star\{\tilde{x},\phi\}_\star\star e^{-1})\Big)\Big]\Big\}.
\end{aligned} \tag{34}
$$

This angular momentum tensor is not conserved, in contrary to the result obtained for the non renormalizable twisted $\phi^{\star 4}$ model studied in [1]. This analysis is compatible with the previous GW model investigation [14]. One recovers the canonical angular momentum tensor of the decoupled fields in the commutative limit. Defining now the dilatation transformation by

$$
x \to x' = \epsilon x; \quad \phi(x) \to \phi'(x') = \phi'(\epsilon x) = \epsilon^{-\Delta}\phi(x), \tag{35}
$$

where ϵ is a constant number, and Δ is the scale dimension of the field ϕ, we note that the GW action is invariant over dilatation symmetry if $\Delta = 0$ and $\epsilon = \pm 1$, implying

$$
x' = x, \quad \phi'(x) = \phi(x); \qquad \text{or} \qquad x' = -x, \quad \phi'(-x) = \phi(x) \tag{36}
$$

which is nothing but a parity transformation of the space-time inducing a conserved current:

$$
\mathcal{D}^\mu = \mathcal{R}^\mu(\delta\tilde{x} \to -2\tilde{x}) - 2x^\mu(\mathcal{L}^\Omega_\star \star e^{-1})e. \tag{37}
$$

Finally, the EMT, AMT and DC can be computed under the well defined field values at the boundary, i.e. $\int e\mathrm{d}^D x\, X_b S(\Delta)(f, \widetilde{X}^b g) = 0$, to give simplified expressions. In this case, there follow

$$
\begin{aligned}
\mathcal{T}^\mu_\nu &= -\frac{e}{2}(\partial_\nu\phi)\{\partial^\mu\phi, e^{-1}\}_\star - \frac{e}{2}(\partial_\nu\phi_c)\{\partial^\mu\phi^c, e^{-1}\}_\star\\
&\quad + e e^\mu_b\Big\{\mathcal{L}^\Omega_\star \star (e^{-1}\partial_\nu\phi^b) + T(\Delta)\Big(X_c\mathcal{L}^\Omega_\star, \widetilde{X}^b(e^{-1}\partial_\nu\phi^c)\Big),
\end{aligned} \tag{38}
$$

and

$$
\begin{aligned}
\mathcal{M}^\mu_{\nu\rho} &= \frac{e}{4}x_{[\nu}\partial_{\rho]}\phi\{\partial^\mu\phi, e^{-1}\}_\star + \frac{e}{4}x_{[\nu}\partial_{\rho]}\phi_c\{\partial^\mu\phi^c, e^{-1}\}_\star\\
&\quad - \frac{e e^\mu_b}{2}\Big(\mathcal{L}^\Omega_\star \star (e^{-1}x_{[\nu}\partial_{\rho]}\phi^b)\Big)\\
&\quad + \frac{e e^\mu_b}{2}\Big\{T(\Delta)\Big(X_c\mathcal{L}^\Omega_\star, \widetilde{X}^b(e^{-1}x_{[\nu}\partial_{\rho]}\phi^c)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-T(\Delta)\Big(\partial_{[\nu}\phi, \frac{1}{2}\widetilde{X}^b(\{\partial_{\rho]}\phi, e^{-1}\}_\star)\Big) \\
&-T(\Delta)\Big(\partial_{[\nu}\phi^d, \frac{1}{2}\widetilde{X}^b(\{\partial_{\rho]}\phi_d, e^{-1}\}_\star)\Big) \\
&-\frac{\Omega^2}{4}\Theta^{-1}_{\gamma[\nu}T(\Delta)\Big(x_{\rho]}, \widetilde{X}^b(\{\phi, \{e^{-1}, \{\tilde{x}^\gamma, \phi\}_\star\}_\star\}_\star)\Big)\Big\} \qquad (39)
\end{aligned}
$$

and the current of dilatation symmetry in the form

$$
\begin{aligned}
\mathcal{D}^\mu \;=\; & -\Omega^2 e e^\mu_b T(\Delta)\Big(\tilde{x}, \widetilde{X}^b\{\phi, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star)\Big) \\
& -2x^\mu(\mathcal{L}^\Omega_\star \star e^{-1})e. \qquad (40)
\end{aligned}
$$

3 Concluding remarks

The following features are worthy of attention from this study:

1. The ordinary ϕ^4-theory leads to nonlocally conserved and symmetric EMT and AMT [6] while the twisted non renormalizable ϕ^4-theory [1] restores the local conservation of these tensors because of nonzero boundary conditions.

2. Both ordinary GW [13, 14] and twisted GW models provide nonlocally conserved and nonsymmetric EMT, AMT and DC due to the presence of the harmonic term Ω.

As shown in [13], all these physical quantities can be subjected to well known Jackiw and Wilson regularization procedures to acquire the local conservation property.

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Appendix

We summarize here useful properties of the dynamical $\star$-product expanded as

$$\begin{aligned} f\star g &= fg+\frac{i}{2}\Theta^{ab}X_afX_bg \\ &\quad +\frac{1}{2!}\left(\frac{i}{2}\right)^2\Theta^{a_1b_1}\Theta^{a_2b_2}(X_{a_1}X_{a_2}f)(X_{b_1}X_{b_2}g)+\cdots \\ &\equiv e^{\Delta}(f,g) \end{aligned} \tag{41}$$

where powers of the bilinear operator Δ are defined as

$$\Delta(f,g)=\frac{i}{2}\Theta^{ab}(X_af)(X_bg), \quad \Delta^0(f,g)=fg$$
$$\Delta^n(f,g)=\left(\frac{i}{2}\right)^n\Theta^{a_1b_1}\cdots\Theta^{a_nb_n}(X_{a_1}\cdots X_{a_n}f)(X_{b_1}\cdots X_{b_n}g). \tag{42}$$

From the definition (41) we deduce the following identities (straightforward generalization of the usual Moyal product identities):

$$f\star g=fg+X_aT(\Delta)(f,\widetilde{X}^ag) \tag{43}$$
$$[f,g]_\star=f\star g-g\star f=2X_aS(\Delta)(f,\widetilde{X}^ag) \tag{44}$$
$$\{f,g\}_\star=f\star g+g\star f=2fg+2X_aR(\Delta)(f,\widetilde{X}^ag) \tag{45}$$

where

$$\begin{aligned} T(\Delta) &= \frac{e^{\Delta}-1}{\Delta}, \quad S(\Delta)=\frac{sinh(\Delta)}{\Delta}, \\ R(\Delta) &= \frac{cosh(\Delta)-1}{\Delta} \text{ and } \widetilde{X}^a=\frac{i}{2}\Theta^{ab}X_b \end{aligned} \tag{46}$$

implying that $S(\Delta)(.,\widetilde{X}.)$ is a bilinear antisymmetric operator and

$$T(\Delta)(f,\widetilde{X}^ag)-T(\Delta)(g,\widetilde{X}^af)=2S(\Delta)(f,\widetilde{X}^ag). \tag{47}$$

The formulas of derivatives and variations are given by [1]

$$\begin{gathered} \delta_{\phi^c}e^\mu_a=-e^\mu_bX_a(\delta\phi^b), \quad \partial_\mu e=eX_a(\partial_\mu\phi^a), \\ \delta_{\phi^c}e=eX_a(\delta\phi^a), \quad \delta_{\phi^c}e^{-1}=-e^{-1}X_a\delta(\phi^a), \\ \delta_{\phi^c}X_a=-X_a(\delta\phi^b)X_b, \quad eX_a(f)=\partial_\mu(ee^\mu_af) \end{gathered} \tag{48}$$

To compute δ_{ϕ^c} variations, the following identity is useful:

$$\delta_{\phi^c}(f\star g) = -(\delta\phi^c X_c f)\star g - f\star(\delta\phi^c X_c g) + \delta\phi^c X_c(f\star g), \tag{49}$$

where the functions f and g do not depend on ϕ^c. By induction, one can immediately prove that (49) holds for $\star$-products of an arbitrary number of factors:

$$\begin{aligned}\delta_{\phi^c}(f\star g\star\cdots\star h) &= -(\delta\phi^c X_c f)\star g\star\cdots\star h\\ &\quad -f\star(\delta\phi^c X_c g)\star\cdots\star h\\ &\quad -\cdots - f\star g\star\cdots\star(\delta\phi^c X_c h)\\ &\quad +\delta\phi^c X_c(f\star g\star\cdots\star h)\end{aligned} \tag{50}$$

One has also:

$$\int \mathrm{d}^D x\,(f\star g) \neq \int \mathrm{d}^D x\,(g\star f), \tag{51}$$

but

$$\int e\mathrm{d}^D x\,(f\star g) = \int e\mathrm{d}^D x(fg) = \int e\mathrm{d}^D x\,(g\star f). \tag{52}$$

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